Sample questions for MATH4060 Midterm Exam

1. State and prove Rouché's theorem, about the comparison of the number of zeroes between two holomorphic functions.

2. Suppose f is holomorphic on $\mathbb{C} \setminus \{0\}$, and has an essential singularity at 0. If $\varepsilon > 0$, let D'_ε be the open punctured disc $\{z \in \mathbb{C} \colon 0 < |z| < \varepsilon\}$. Show that for any $\varepsilon > 0$, the image under f of D'_ε is dense in $\mathbb{C}$.

3. Weierstrasss theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc in $\mathbb{C}$ be approximated uniformly by polynomials in the variable z?

4. Let $\mathbb{D}$ be the open unit disc $\{z \in \mathbb{C} \colon |z| < 1\}$. Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \quad \text{whenever } |z| = 1,$$

then f is constant.

5. Let $a > 0$, and S be the strip $\{z \in \mathbb{C} \colon |\operatorname{Im} z| < a\}$. Suppose f is holomorphic on S, and there exists a constant A such that

$$|f(x + iy)| \leq \frac{A}{1 + x^2} \quad \text{for all } x + iy \in S.$$

Show that for any $b \in [0, a)$, there exists a constant B such that the Fourier transform $\widehat{f}$ of f satisfies

$$|\widehat{f}(\xi)| \leq B e^{-2\pi b|\xi|} \quad \text{for all } \xi \in \mathbb{R}.$$

6. Show that for any $z \in \mathbb{C}$ that is not an integer, we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z + n)^2} = \frac{\pi^2}{\sin^2(\pi z)}.$$

Hence, or otherwise, deduce a factorization of $\sin(\pi z)$ into an infinite product.

7. (a) Suppose $\{f_n\}$ is a sequence of functions on a domain Ω, and f_n converges uniformly to some function f on Ω. If $\{f_n\}$ is uniformly bounded, in the sense that there exists a constant M such that $|f_n(z)| \leq M$ for all $n \in \mathbb{N}$ and all $z \in \Omega$, then e^{f_n} converges uniformly to e^f on Ω.

   (b) Suppose $\{g_n\}$ is a sequence of functions on a domain Ω, and there exists a sequence of non-negative numbers $\{c_n\}$, with $\sum_{n=1}^{\infty} c_n < \infty$, such that

$$|g_n(z) - 1| \leq c_n \quad \text{for all } n \in \mathbb{N} \text{ and all } z \in \Omega.$$

   Using (a), or otherwise, show that $\prod_{j=1}^{n} g_j$ converges uniformly on Ω as $n \to \infty$.

8. Find the Hadamard factorization of the function $\cosh z - 1$.

9. Show that if f is an entire function of finite order that omits two values, then f is constant.

10. Show that the equation $e^z - z = 0$ has infinitely many solutions in $\mathbb{C}$.